1.2: Numerical Integration

- .Let $f : [a, b] \to \Re$ be a continuous function.
- The aim is to compute the integral:

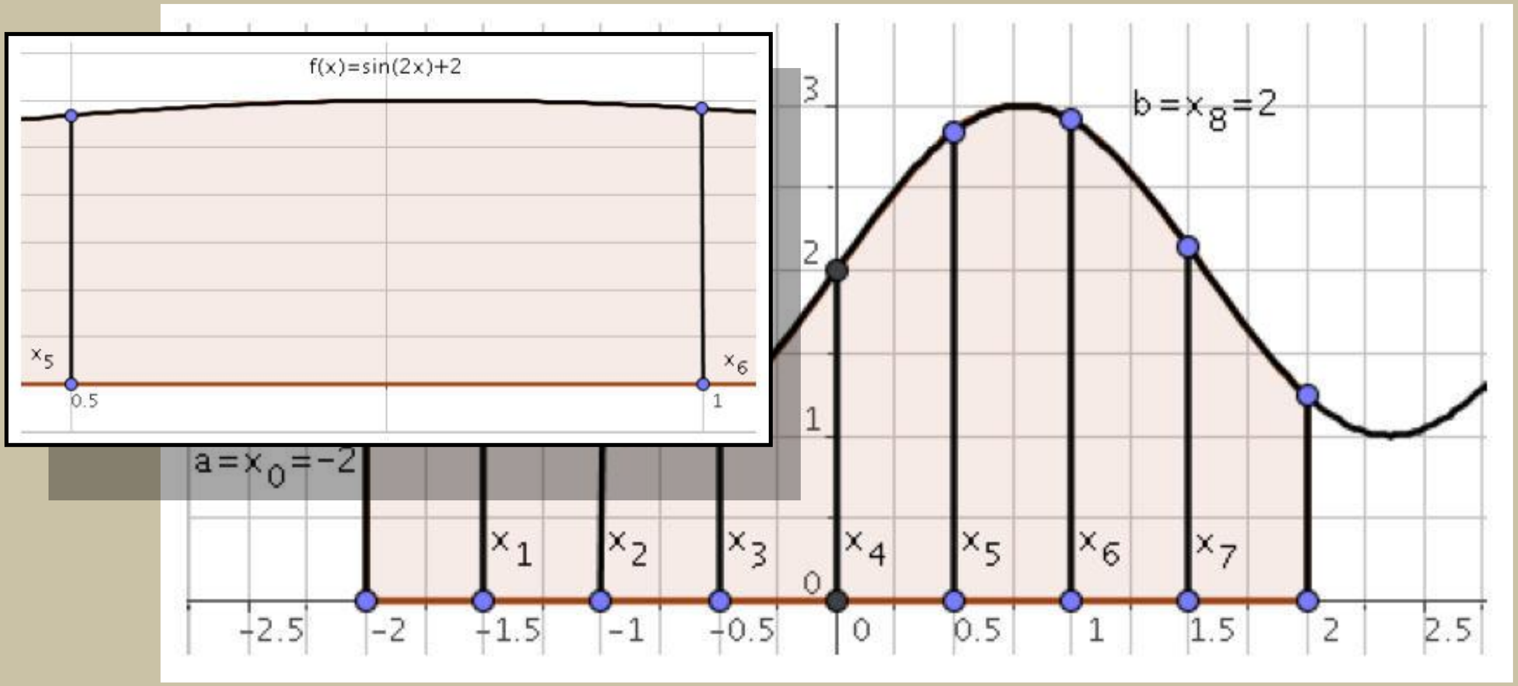
$$I(f) = \int_a^b f(x)dx$$

by a finite sum:

$$I(f) \approx \sum_i w_i f(x_i)$$

Interval partitioning:

- $[a,b]$: $[x_i, x_{i+1}]$, $i = 0,\ldots,n-1$ with $a = x_0 < \ldots < x_n = b$



$$I(f) = \int_a^b f(x)dx = \sum_{i=0}^{i=n-1} \int_{x_i}^{x_{i+1}} f(x)dx$$

Change of interval:

- x: [x_i, x_{i+1}] → t: [-1, 1]: $t = 2\dfrac{x - x_i}{x_{i+1} - x_i} - 1$

$$\int_{x_i}^{x_{i+1}} f(x)dx = \frac{x_{i+1} - x_i}{2} \int_{-1}^{1} g_i(t)dt$$

$$g_i(t) = f\left(\frac{x_{i+1} - x_i}{2}(t+1) + x_i\right), \quad i = 0, \ldots, n-1.$$

- Can we evaluate the integral *exactly* by a finite sum?

Quadrature rule:

$$\int_{-1}^{1} g(t)dt \approx \sum_{j=0}^{p} \omega_j g(t_j)$$

- -1 ≤ t_0 <...< t_p ≤ 1: points of integration (nodes).
- ω_j : weights
- Let $Q \in P_Q$ be the set of polynomials of degree at most equal to q. Then the quadrature rule is said to be *exact* in P_Q if,

$$\forall Q \in P_Q, \int_{-1}^{1} Q(t)dt = \sum_{j=0}^{p} \omega_j Q(t_j)$$

Quadrature rule in Lagrange basis:

- Let $L_0, L_1, \ldots L_p$ be the Lagrange basis of P relative to the points $-1 \le t_0 < \ldots < t_p \le 1$. Then

$$\forall Q \in P_Q, \int_{-1}^{1} Q(t)\,dt = \sum_{j=0}^{p} \omega_j Q(t_j)$$

is exact for

$$\omega_j = \int_{-1}^{1} L_j(t)\,dt, \quad j = 0,1,\ldots, p.$$

Newton-Cotes Formula:

- $a = x_0 < \ldots < x_n = b$ the points are equidistant: $x_i = x_0 + hi$, $h = (b-a)/n$

$$\int_{x_i}^{x_{i+1}} f(x)dx = \frac{x_{i+1} - x_i}{2} \int_{-1}^{1} g_i(t)dt$$

$$\approx \frac{h}{2} \sum_{j=0}^{p} \omega_j g_i(t_j)$$

$$\approx \frac{h}{2} \sum_{j=0}^{p} \omega_j f\left(\frac{h}{2}(t_j + 1) + x_i \right)$$

Newton-Cotes Formula:

$$I(f) = \int_a^b f(x)dx \quad = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} f(x)dx$$

$$\approx \frac{h}{2} \sum_{i=0}^{n-1} \sum_{j=0}^{p} \omega_j f\left(\frac{h}{2}(t_j + 1) + x_i \right)$$

- p = 0 method of rectangles
- p = 1 method of trapezes
- p = 2 Simpson's method
- p = 3 Simpson's 3/8 method
- p = 4 Bode's method

Trapezoid Rule:

- $p = 1 \rightarrow t_0 = -1,\ t_1 = 1$. Lagrange basis for P_1:
  $L_0 = (1 - t)/2,\ L_1 = (1 + t)/2$
- weights:

$$\omega_0 = \int_{-1}^{1} L_0(t)\,dt = 1; \quad \omega_1 = \int_{-1}^{1} L_1(t)\,dt = 1$$

- Rule:

$$\int_{-1}^{1} g(t)\,dt \approx \sum_{j=0}^{p} \omega_j g(t_j) = g(-1) + g(1)$$

Trapezoid Rule:

$$I(f) = \int_a^b f(x)dx \quad = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} f(x)dx$$

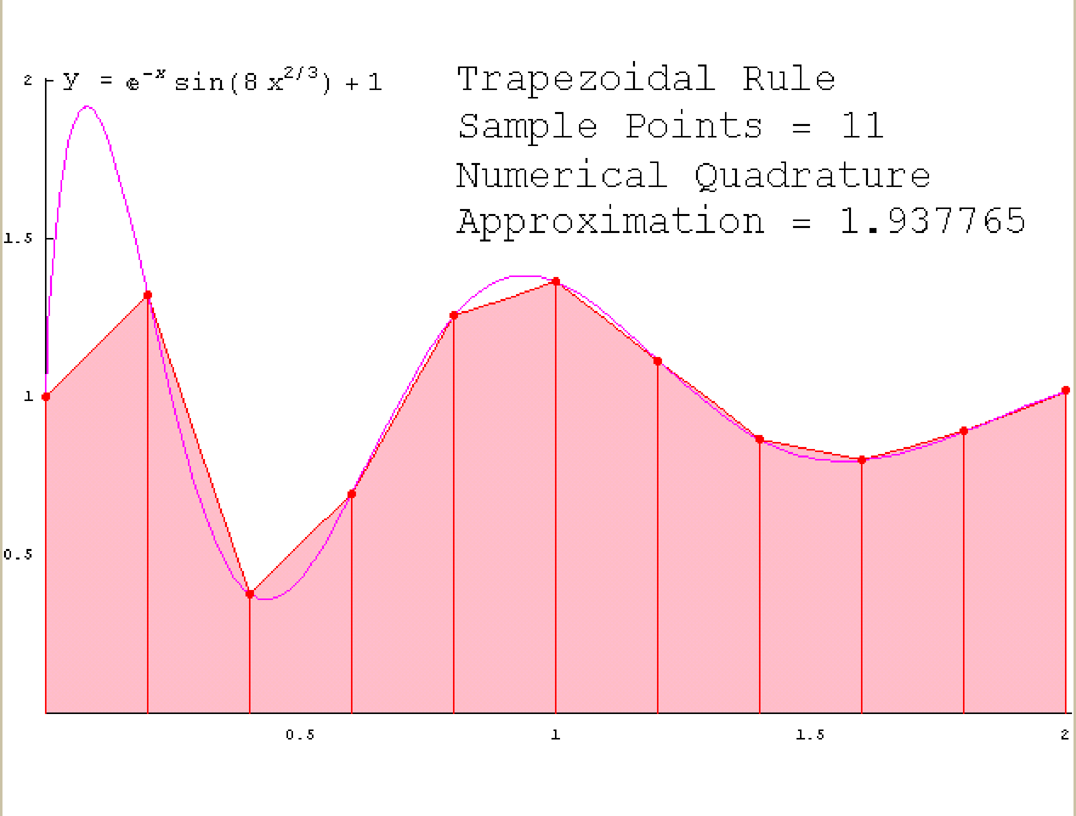
$$\approx \frac{h}{2} \sum_{i=0}^{n-1} \sum_{j=0}^{p} \omega_j f\left(\frac{h}{2}(t_j + 1) + x_i \right)$$

$$\approx \frac{h}{2} \sum_{i=0}^{n-1} \left[f(x_i) + f(x_{i+1}) \right]$$

Trapezoid Rule:

$$I(f) = \int_a^b f(x)dx \approx \frac{h}{2}\sum_{i=0}^{n-1}\left(f(x_i) + f(x_{i+1})\right)$$

We interpolate the function at the two endpoints of the interval by a straight line (i.e. polynomial of degree one).



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Simpson's Rule:

- $p = 2 \rightarrow t_0 = -1$, $t_1 = 0$, $t_2 = 0$. Lagrange basis for P_2:

  $L_0 = (t^2 - t)/2$, $L_1 = 1 - t^2$, $L_2 = (t^2 + t)/2$
- weights:

$$\omega_0 = \int_{-1}^{1} L_0(t)dt = \frac{1}{3}; \;\; \omega_1 = \int_{-1}^{1} L_1(t)dt = \frac{4}{3}; \;\; \omega_2 = \int_{-1}^{1} L_2(t)dt = \frac{1}{3}$$

- Rule:

$$\int_{-1}^{1} g(t)dt \approx \sum_{j=0}^{p} \omega_j g(t_j) = \frac{1}{3} g(-1) + \frac{4}{3} g(0) + \frac{1}{3} g(1)$$

Simpson's Rule:

$$I(f) = \int_a^b f(x)dx \quad = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} f(x)dx$$

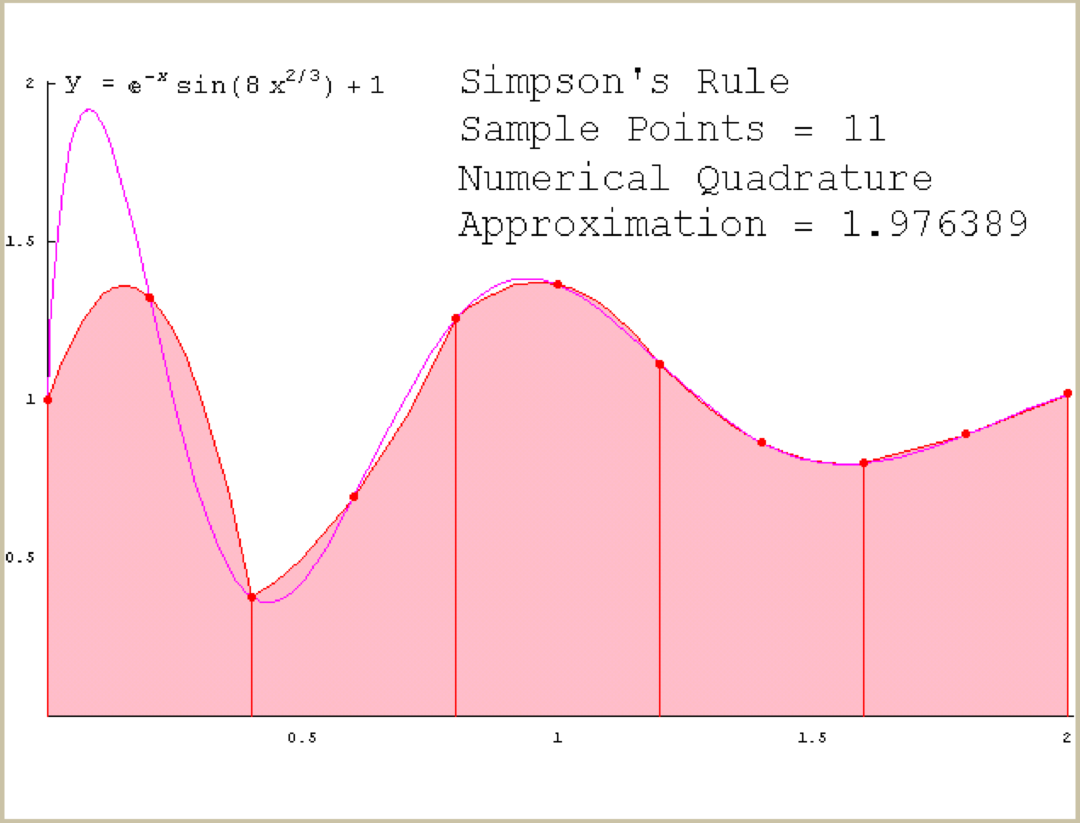
$$\approx \frac{h}{2} \sum_{i=0}^{n-1} \sum_{j=0}^{p} \omega_j f\left(\frac{h}{2}(t_j + 1) + x_i \right)$$

$$\approx \frac{h}{6} \sum_{i=0}^{n-1} \left(f(x_i) + 4 f(\frac{x_i + x_{i+1}}{2}) + f(x_{i+1}) \right)$$

Simpson's Rule:

$$I(f)=\int_a^b f(x)dx \approx \frac{h}{6}\sum_{i=0}^{n-1}\left(f(x_i)+4f(\frac{x_i+x_{i+1}}{2})+f(x_{i+1})\right)$$

We interpolate the function at three points (two endpoints and the middle) by a quadratic polynomial.



Summary:

- Midpoint rule:

$$I(f)=\int_a^b f(x)dx \approx h\sum_{i=0}^{n-1} f\left(\frac{x_i+x_{i+1}}{2}\right)$$

- Trapezoid rule:

$$I(f)=\int_a^b f(x)dx \approx \frac{h}{2}\sum_{i=0}^{n-1}\left[f(x_i)+f(x_{i+1})\right]$$

- Simpson's rule:

$$I(f)=\int_a^b f(x)dx \approx \frac{h}{6}\sum_{i=0}^{n-1}\left(f(x_i)+4f\left(\frac{x_i+x_{i+1}}{2}\right)+f(x_{i+1})\right)$$